

Categorification of Clifford algebra via geometric induction and restriction

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INTRODUCTION

In a book published in 1981 ([7]), Andrei Zelevinsky categorified an infinite-rank PSH-algebra in terms of representations of the collection of all $GL(n, \mathbb{F})$ where \mathbb{F} is a finite field. He did this using a pair of adjoint functors, the parabolic induction and its adjoint.

We intend, in this paper, to apply the same set of ideas to the categorification of a infinite Clifford algebra acting on the Fock space of semi-infinite forms, in terms of representations of the collection of all classical supergroups $SOSP(2m+1, 2n)$, using the geometric induction functor and its adjoint called geometric restriction.

Let us start with the preliminary example of classical groups.

Let $(G_n)_{n \geq 1}$ be a family of complex classical Lie groups, G_n of rank n , together with inclusions

$$G_1 \subset G_2 \subset \dots \subset G_n \subset \dots$$

in such a way that $G_{n-1} \times \mathbb{C}^*$ is the reductive part of a maximal parabolic subgroup denoted P_n of G_n , and we denote the maximal unipotent subgroup of P_n by U_n . For instance, consider $G_n = GL(n, \mathbb{C})$. We use gothic letters for the corresponding Lie algebras. We denote \mathcal{F}_n the category of finite-dimensional G_n -modules, it is a semi-simple category and we denote \mathcal{K}_n its Grothendieck group.

We use the functors Γ_i^a and H_b^j defined as follows:

$$\Gamma_i^a : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1},$$

$$\Gamma_i^a(M) := H^i(G_{n+1}/P_{n+1}, \mathcal{L}(\mathbb{C}_{a+n} \boxtimes M)^*)^*$$

where \mathbb{C}_a is the one-dimensional representation of \mathbb{C}^* with character $a \in \mathbb{Z}$; we assume that U_n acts trivially and $\mathcal{L}(\mathbb{C}_a \boxtimes M)^*$ is the induced vector bundle $G_n \times_{P_{n+1}} (\mathbb{C}_a \boxtimes M)^*$.

$$H_b^j : \mathcal{F}_n \rightarrow \mathcal{F}_{n-1},$$

$$H_b^j(M) := \text{Hom}_{\mathbb{C}^*}(\mathbb{C}_{b+n}, H^j(\mathfrak{u}_n, M)).$$

At the level of Grothendieck groups we obtain linear maps

$$\gamma^a : \mathcal{K}_n \rightarrow \mathcal{K}_{n+1}$$

$$[M] \mapsto \sum_i (-1)^i [H^i(G_{n+1}/P_{n+1}, \mathcal{L}(\mathbb{C}_{a+n} \boxtimes M)^*)^*],$$

$$\eta_b : \mathcal{K}_n \rightarrow \mathcal{K}_{n-1}$$

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$$[M] \mapsto \sum_j (-1)^j [\text{Hom}_{\mathbf{C}^*}(\mathbb{C}_{b+n}, H^j(\mathbf{u}_n, M))].$$

We set $\mathcal{K} := \oplus_n \mathcal{K}_n$ and extend those maps to \mathcal{K} . Then, applying Borel-Weil-Bott theorem, we obtain the following relations, for all a and b in \mathbb{Z} :

$$(1) \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 0,$$

$$(2) \quad \eta_a \eta_b + \eta_b \eta_a = 0,$$

$$(3) \quad \gamma^a \eta_b + \eta_b \gamma^a = \delta_{a,b} Id.$$

We recognise those relations as the ones of the infinite dimensional Clifford algebra \mathbf{C} . Furthermore, we see \mathcal{K} as an irreducible representation of \mathbf{C} which is induced by the trivial representation of the subalgebra of \mathbf{C} generated by $(\eta_b)_{b \in \mathbb{Z}}$.

This provides a categorification of the Clifford algebra \mathbf{C} by the family of classical groups $(G_n)_{n \geq 1}$.

We follow the same scheme for the family of classical Lie supergroups $SOSP(2m+1, 2n)$ when m and n vary, in this case we categorify the representation of the infinite Clifford algebra in the Fock space of semi-infinite forms. In the last section, we explain how our previous categorification work on orthosymplectic Lie superalgebras ([3]) can be understood in this context.

We would also like to mention the work of Michael Ehrig and Catharina Stroppel [1], who used quantized symmetric pairs in order to refine our previous results on the category of finite dimensional modules over orthosymplectic Lie superalgebras and obtain a diagrammatic description of the endomorphism algebras of projective generators.

It would be very interesting now to construct a canonical basis in the Fock space of semi-infinite forms.

Finally, we would like to emphasize that what we do here can easily be done for all series of classical Lie supergroups, with minor changes only.

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1. BASIC SETTING

We work over the field of complex numbers in the category of $\mathbb{Z}/2\mathbb{Z}$ -graded spaces. The reader should keep it in mind when we consider symmetric and exterior powers.

We denote by $\mathfrak{g}_{m,n}$ the Lie superalgebra $\mathfrak{osp}(2m+1, 2n)$ and

$$\mathfrak{g}_{\infty, \infty} = \varinjlim_{m, n \rightarrow \infty} \mathfrak{g}_{m, n}.$$

Further more, we fix an embedding $\mathfrak{g}_{m,n} \subset \mathfrak{g}_{\infty, \infty}$.

We also fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\infty, \infty}$ and the standard basis $\{\varepsilon_i, \delta_j\}_{i, j \in \mathbb{Z}_{>0}}$. The roots of $\mathfrak{g}_{\infty, \infty}$ in this basis are:

$$(\pm\varepsilon_i), \quad (\pm\delta_j),$$

$$(\pm\varepsilon_i \pm \delta_j), \quad (\pm 2\delta_j),$$

$$(\pm\varepsilon_i \pm \varepsilon_j), \quad (\pm\delta_i \pm \delta_j),$$

where i, j vary from 1 to ∞ , and in the last line, $i \neq j$.

Then the roots of $\mathfrak{g}_{m, n}$ lie in the subspace generated by $(\varepsilon_i)_{1 \leq i \leq m}$ and $(\delta_j)_{1 \leq j \leq n}$.

We fix a Borel subalgebra \mathfrak{b}_0 of $(\mathfrak{g}_{\infty, \infty})_0$ with the set of positive roots

$$\{\varepsilon_i, 2\delta_j, (i, j > 0), \varepsilon_i \pm \varepsilon_j, \delta_i \pm \delta_j (i > j > 0)\}.$$

Inside $\mathfrak{g}_{m, n}$, we denote by $\mathfrak{p}_{m, n}$ (resp $\mathfrak{p}_{m, n}$) the unique parabolic subalgebra containing \mathfrak{b}_0 with semi-simple part $\mathfrak{g}_{m-1, n}$ (resp $\mathfrak{g}_{m, n-1}$).

We denote by $G_{m, n}$ the supergroup $SOSP(2m+1, 2n)$ and by $T_{m, n}$ the maximal torus of $G_{m, n}$ with Lie algebra $\mathfrak{h} \cap \mathfrak{g}_{m, n}$.

For fixed m and n , we denote by $\mathcal{F}_{m, n}$ the category of finite dimensional $G_{m, n}$ -modules and by $\mathcal{K}_{m, n}$ its Grothendieck group.

Let: $\mathcal{F} := \oplus_{m, n} \mathcal{F}_{m, n}$ and $\mathcal{K} := \oplus_{m, n} \mathcal{K}_{m, n}$.

Let B be a Borel subgroup of $G_{m, n}$ with Lie superalgebra \mathfrak{b} containing $\mathfrak{b}_0 \cap \mathfrak{g}_{m, n}$ and let Δ_1^+ (resp. Δ_0^+) be the set of odd (resp.) even positive roots of $\mathfrak{g}_{m, n}$. Set

$$\rho_B = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha.$$

Let $\Lambda_{m, n}$ be the set of weights λ such that $\lambda - \rho_B$ is a character of $T_{m, n}$. Independently of the choice of B we have

$$\Lambda_{m, n} := \{\lambda = a_1\varepsilon_1 + \cdots + a_m\varepsilon_m + b_1\delta_1 + \cdots + b_n\delta_n \mid a_i, b_j \in \frac{1}{2} + \mathbb{Z}\}.$$

We set

$$\Lambda_{m, n}^+ := \{\lambda \in \Lambda_{m, n} \mid a_i, b_j \in \frac{1}{2} + \mathbb{N}, a_1 < \cdots < a_m, b_1 < \cdots < b_n\}.$$

Let ν be a character of $T_{m, n}$. We denote by \mathcal{L}_ν the corresponding line bundle over $G_{m, n}/B$.

Recall the definition of the *Euler characteristic*. For every $\lambda \in \Lambda_{m, n}$ we set

$$\mathcal{E}(\lambda) := \left[\sum_{i \geq 0} (-1)^i H^i(G_{m, n}/B, \mathcal{L}_{\lambda - \rho_B}^*)^* \right] \in \mathcal{K}_{m, n}.$$

Recall also that the character of this virtual module is easy to compute, namely

$$Ch(\mathcal{E}(\lambda)) = \frac{D_0}{D_1} \sum_{w \in W_{m, n}} \varepsilon(w) e^{w(\lambda)},$$

where $W_{m,n}$ is the Weyl group of $SO(2m+1) \times SP(2n)$, $D_0 = \Pi_{\alpha \in \Delta_0^+}(e^{\alpha/2} - e^{-\alpha/2})$, $D_1 = \Pi_{\alpha \in \Delta_1^+}(e^{\alpha/2} + e^{-\alpha/2})$.

(Remark: We avoided indexes m and n in this formula since one can easily recover them from the shape of λ).

Note that if we change our choice of B containing $B_0 \cap G_{m,n}$, the character of $\mathcal{E}(\lambda)$ doesn't change, thus the class in $\mathcal{K}_{m,n}$ remains the same, see [3].

For $w \in W_{m,n}$, notice that

$$(4) \quad \mathcal{E}(w(\lambda)) = \varepsilon(w)\mathcal{E}(\lambda).$$

Proposition 1. (see [3]) *The set*

$$\{\mathcal{E}(\lambda), \lambda \in \Lambda_{m,n}^+\}$$

gives a linearly independant family in $\mathcal{K}_{m,n}$, and we denote by $\mathcal{K}(\mathcal{E})_{m,n}$ the subgroup generated by this family. We also set $\mathcal{K}(\mathcal{E}) := \oplus_{m,n} \mathcal{K}(\mathcal{E})_{m,n}$.

2. FOCK SPACE

Let V be a countable dimensional vector space together with a basis $(v_i)_{i \in \frac{1}{2} + \mathbb{Z}}$ and similarly W with a basis $(w_i)_{i \in \frac{1}{2} + \mathbb{Z}}$ with a non-degenerate pairing such that (v_i) and (w_i) are dual bases.

Let $Cl(V \oplus W)$ be the *Clifford algebra* of $V \oplus W$, namely if we denote by $T(V \oplus W)$ the tensor algebra of $V \oplus W$,

$$Cl(V \oplus W) = T(V \oplus W) / (v \otimes v' + v' \otimes v, w \otimes w' + w' \otimes w, v \otimes w + w \otimes v - (v, w), v, v' \in V, w, w' \in W).$$

The *Fock space* of semi-infinite forms, \mathbf{F} , is the vector space generated by

$$v_{i_1} \wedge \dots \wedge v_{i_k} \wedge \dots,$$

for $i_1 > \dots > i_k > \dots$ such that, for n large enough, $i_n = i_{n-1} - 1$.

There is a natural linear action of $Cl(V \oplus W)$ on \mathbf{F} given by:

$$\begin{aligned} \forall v \in V, v \bullet v_{i_1} \wedge \dots \wedge v_{i_k} \wedge v_{i_{k+1}} \dots &= v \wedge v_{i_1} \wedge \dots \wedge v_{i_k} \wedge v_{i_{k+1}} \dots \\ \forall w \in W, w \bullet v_{i_1} \wedge \dots \wedge v_{i_k} \wedge v_{i_{k+1}} \dots &= \sum_j (-1)^{j-1} (w, v_{i_j}) v_{i_1} \wedge \dots \wedge \hat{v}_{i_j} \wedge \dots \end{aligned}$$

Define the *vacuum vector* in \mathbf{F} as

$$|> := v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge \dots$$

then, for $i < 0$, v_i acts on $|>$ by 0 as w_j for $j > 0$.

We can also see \mathbf{F} as an induced module the following way. Denote by $Cl^+(V \oplus W)$ the subalgebra generated by $\{v_i, i < 0, w_j, j > 0\}$, consider its trivial module and induce to the whole $Cl(V \oplus W)$: this gives another construction of \mathbf{F} .

Let $\lambda = \sum_{1 \leq i \leq m, 1 \leq j \leq n} a_i \varepsilon_i + b_j \delta_j \in \Lambda_{m,n}^+$. We define a \mathbb{Z} -linear map $f: \mathcal{K}(\mathcal{E}) \rightarrow \mathbf{F}$ such that for any $\mathcal{E}(\lambda) \in \mathcal{K}(\mathcal{E})_{m,n}$:

$$\mathcal{E}(\lambda) \mapsto v_{a_m} \wedge \dots \wedge v_{a_1} \wedge \dots \wedge \hat{v}_{-b_1} \wedge \dots \wedge \hat{v}_{-b_n} \wedge \dots$$

3. DUALITY BETWEEN GEOMETRIC INDUCTION AND RESTRICTION

In this section we will consider 3 different Grothendieck groups for $G_{m,n}$ namely $\mathcal{K}(P)_{m,n}$ generated by the indecomposable projective modules, $\mathcal{K}(\mathcal{E})_{m,n}$ which we already met and $\mathcal{K}(L)_{m,n} := \mathcal{K}_{m,n}$ generated by the simple modules. After tensoring by the rational numbers \mathbb{Q} , $\mathcal{K}(P)_{m,n} \otimes \mathbb{Q}$ and $\mathcal{K}(\mathcal{E})_{m,n} \otimes \mathbb{Q}$ coincide (see [3]). We consider the natural pairing between $\mathcal{K}(P)_{m,n}$ and $\mathcal{K}(L)_{m,n}$, $\langle [P], [L] \rangle := \dim \operatorname{Hom}(P, L)$. The restriction of this pairing to $\mathcal{K}(P)_{m,n} \times \mathcal{K}(P)_{m,n}$ is symmetric (and therefore it is a scalar product): indeed $\dim \operatorname{Hom}(P_1, P_2) = \dim \operatorname{Hom}(P_2^*, P_1^*)$ and in this case projective modules happen to be self-dual (see [6]).

Proposition 2. *Let us extend the scalar product from $\mathcal{K}(P)_{m,n}$ to $\mathcal{K}(P)_{m,n} \otimes \mathbb{Q}$. Then the set of $\mathcal{E}(\lambda)$, when λ varies in $\Lambda_{m,n}^+$, form an orthonormal basis of $\mathcal{K}(P)_{m,n} \otimes \mathbb{Q}$.*

Proof. Let $L(\lambda)$ denote the simple module with highest weight λ and $P(\lambda)$ denote its projective cover. Consider the decompositions

$$[P(\lambda)] = \sum_{\mu} b_{\lambda,\mu} \mathcal{E}(\mu), \quad \mathcal{E}(\mu) = \sum_{\nu} a_{\mu,\nu} [L(\nu)].$$

By the weak BGG reciprocity, [3], we have $b_{\lambda,\mu} = a_{\mu,\lambda}$. Now, we write

$$\mathcal{E}(\mu) = \sum_{\lambda} c_{\mu,\lambda} [P(\lambda)].$$

Then, clearly, we have the following relation

$$\sum_{\lambda} c_{\mu,\lambda} b_{\lambda,\nu} = \sum_{\lambda} c_{\mu,\lambda} a_{\nu,\lambda} = \delta_{\mu,\nu}.$$

On the other hand,

$$\langle [P(\lambda)], [L(\kappa)] \rangle = \delta_{\lambda,\kappa}.$$

Therefore

$$\langle \mathcal{E}(\mu), \mathcal{E}(\nu) \rangle = \sum_{\lambda, \kappa} c_{\mu,\lambda} a_{\nu,\kappa} \langle [P(\lambda)], [L(\kappa)] \rangle = \sum_{\lambda} c_{\mu,\lambda} a_{\nu,\lambda} = \delta_{\mu,\nu}.$$

□

Let G be a quasireductive algebraic supergroup, which is an algebraic supergroup with reductive even part (see [6] for information on their representation theory). Let $Q \subset G$ be a parabolic subgroup with quasireductive part K . Let $\mathfrak{g}, \mathfrak{q}, \mathfrak{k}$ denote the respective Lie superalgebras, and let \mathfrak{r} denote the nil-radical of \mathfrak{q} . Consider the following derived functors $\Gamma_i : K\text{-mod} \rightarrow G\text{-mod}$ and $H^i : G\text{-mod} \rightarrow K\text{-mod}$ defined by

$$\Gamma_i(M) := H^i(G/Q, \mathcal{L}(M^*))^*, \quad H^i(N) := H^i(\mathfrak{r}, N).$$

Here we denote by $\mathcal{L}(M^*)$ the vector bundle on G/Q induced from M^* . The collection of functors Γ_i is referred to as *geometric induction* while that of H^i is referred to as *geometric restriction*.

The following observation is due to Penkov [5].

Proposition 3. *For any K -module M we have*

$$\sum_i (-1)^i [\Gamma_i(M)] = \sum_i (-1)^i [H^i(G_0/Q_0, \mathcal{L}(S^\bullet(\mathfrak{r}) \otimes M^*))^*].$$

Proposition 4. *For every projective G -module P , every K -module M and $i \geq 0$ there is a canonical isomorphism*

$$\mathrm{Hom}_G(\Gamma_i(M), P) \simeq \mathrm{Hom}_K(M, H^i(P)).$$

Proof. This result is a slight generalization of Proposition 1 in [3]. We consider an injective resolution $0 \rightarrow R^0 \rightarrow R^1 \rightarrow \dots$ of M in the category of Q -modules. Since $\mathrm{Hom}_G(P, \cdot)$ is an exact functor, $\mathrm{Hom}_G(P, H^i(G/Q, M))$ is given by the i -th cohomology group of the complex

$$0 \rightarrow \mathrm{Hom}_G(P, H^0(G/Q, R^0)) \rightarrow \mathrm{Hom}_G(P, H^0(G/Q, R^1)) \rightarrow \dots$$

The Frobenius reciprocity implies

$$\mathrm{Hom}_G(P, H^0(G/Q, R^j)) \simeq \mathrm{Hom}_Q(P, R^j).$$

Thus, we obtain the isomorphism

$$\mathrm{Hom}_G(P, H^i(G/Q, M)) \simeq \mathrm{Ext}_Q^i(P, M).$$

We now need the following lemma.

Lemma 1. *The restricted module $\mathrm{Res}_K P$ is projective in the category $K - \mathrm{mod}$.*

Proof. Note that P is a direct summand of the induced module $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} S$ for some semisimple \mathfrak{g}_0 -module S . Using the isomorphism

$$\mathrm{Res}_K \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} S \simeq \mathrm{Ind}_{\mathfrak{k}_0}^{\mathfrak{k}} S \otimes S^\bullet(\mathfrak{g}_1/\mathfrak{k}_1),$$

we obtain that P is a direct summand of some module induced from a semisimple \mathfrak{k}_0 -module. Therefore P is projective as a K -module. \square

Applying the above lemma we can use the Koszul complex $\Lambda^i(\mathfrak{r}) \otimes \mathcal{U}(\mathfrak{r}) \otimes P$ (where $\mathcal{U}(\mathfrak{r})$ is the universal enveloping algebra of \mathfrak{r}) and thus obtain an isomorphism

$$\mathrm{Ext}_Q^i(P, M) \simeq \mathrm{Hom}_K(H_i(\mathfrak{r}, P), M).$$

Now we use the double dualization and the fact that P^* is also projective:

$$\begin{aligned} \mathrm{Hom}_G(\Gamma_i(M), P) &\simeq \mathrm{Hom}_G(P^*, H^i(G/Q, M^*)) \simeq \mathrm{Hom}_K(H_i(\mathfrak{r}, P^*), M^*) \simeq \\ &\simeq \mathrm{Hom}_K(M, H^i(P)). \end{aligned}$$

Hence the statement. \square

Recall that for any quasireductive supergroup every projective module is injective and vice versa, [6].

Corollary 1. *If P is an injective (equivalently, projective) G -module, then $H^i(P)$ is an injective and projective K -module.*

4. TWO FUNCTORS ON \mathcal{F}

We choose a parabolic subalgebra \mathfrak{p} which can be either $\mathfrak{p}_{\underline{m},n}$ or $\mathfrak{p}_{m,\underline{n}}$ in $\mathfrak{g}_{m,n}$, where:

$$\begin{aligned}\mathfrak{p}_{\underline{m},n} &= \mathfrak{g}_{m-1,n} \oplus \mathbb{C}z \oplus \mathfrak{r}_{\underline{m},n}, \quad \mathfrak{g}_{m,n} = \mathfrak{p}_{\underline{m},n} \oplus \mathfrak{r}_{\underline{m},n}^- \\ \mathfrak{p}_{m,\underline{n}} &= \mathfrak{g}_{m,n-1} \oplus \mathbb{C}z \oplus \mathfrak{r}_{m,\underline{n}}, \quad \text{and } \mathfrak{g}_{m,n} = \mathfrak{p}_{m,\underline{n}} \oplus \mathfrak{r}_{m,\underline{n}}^-.\end{aligned}$$

Denote by Z the center of the reductive part of the parabolic subgroup P corresponding to the parabolic subalgebra we chose above (the Lie algebra of Z is $\mathbb{C}z$). For any $a \in \mathbb{Z}$ we denote by \mathbb{C}_a the corresponding character of Z . Since Z is a one-parameter subgroup of the torus $T_{m,n}$, if ϖ is the corresponding weight in the dual, we denote by $\mathbb{C}_{a\varpi}$ the associated $T_{m,n}$ -module (in our case, ϖ is either ε_n or δ_n). Now, if $M \in \mathcal{F}_{m-1,n}$ or $\mathcal{F}_{m,n-1}$, denote $\mathbb{C}_a \boxtimes M$ the P -module with trivial action of the corresponding nilradical \mathfrak{R} and the given action of $Z \times G_{m-1,n}$, or $Z \times G_{m,n-1}$ depending on the way the parabolic is chosen.

Definition 1. *We define the following functors:*

$$\Gamma_i^a : \mathcal{F} \rightarrow \mathcal{F}, a \in \frac{1}{2} + \mathbb{Z}$$

if $a > 0$, if $M \in \mathcal{F}_{m-1,n}$, $\Gamma_i^a(M) := H^i(G_{m,n}/P_{\underline{m},n}, \mathcal{L}(\mathbb{C}_{(a-(m-n-\frac{1}{2}))\varepsilon_m} \boxtimes M)^*)^*$,
if $a < 0$, if $M \in \mathcal{F}_{m,n-1}$, $\Gamma_i^a(M) := H^i(G_{m,n}/P_{m,\underline{n}}, \mathcal{L}(\mathbb{C}_{(-a-(n-m-\frac{1}{2}))\delta_n} \boxtimes M)^*)^*$.

$$H_b^j : \mathcal{F} \rightarrow \mathcal{F}, b \in \frac{1}{2} + \mathbb{Z}$$

if $b > 0$, if $M \in \mathcal{F}_{m,n}$, $H_b^j(M) := \text{Hom}_Z(\mathbb{C}_{(b-(m-n-\frac{1}{2}))\varepsilon_m}, H^j(\mathfrak{r}_{\underline{m},n}, M)) \in \mathcal{F}_{m-1,n}$
if $b < 0$, if $M \in \mathcal{F}_{m,n}$, $H_b^j(M) := \text{Hom}_Z(\mathbb{C}_{(-b-(n-m-\frac{1}{2}))\delta_n}, H^j(\mathfrak{r}_{m,\underline{n}}, M)) \in \mathcal{F}_{m,n-1}$.

Now, we consider the following operators in \mathcal{K} : if $M \in \mathcal{F}_{m,n}$, denoting the sign of a half-integer x by $\text{sgn}(x)$:

$$\gamma^a([M]) := \text{sgn}(a)^m \sum_{i \geq 0} (-1)^i [\Gamma_i^a(M)]$$

$$\eta_b([M]) := \text{sgn}(b)^m \sum_{j \geq 0} (-1)^j [H_b^j(M)].$$

Applying the results of the previous section, we get:

Proposition 5. *Consider the pairing $\mathcal{K}(L) \times \mathcal{K}(P) \rightarrow \mathbb{Z}$ defined by*

$$\langle [M], [P] \rangle := \dim \text{Hom}_{G_{m,n}}(M, P)$$

for every projective $P \in \mathcal{F}_{m,n}$ and every $M \in \mathcal{F}_{m,n}$. Then for any $a \in \frac{1}{2} + \mathbb{Z}$ we have

$$\langle M, \eta_a[P] \rangle = \langle \gamma^a[M], [P] \rangle.$$

Let us restrict those linear operators to $\mathcal{K}(\mathcal{E})$. Then for every $a \in \frac{1}{2} + \mathbb{Z}$, the linear operators γ^a and η_a are mutually adjoint.

We can identify the Grothendieck ring with the ring of characters of finite dimensional modules (cf [3]) and so we will check the relations we need at the level of characters.

We recall the following formula ([2], prop. 1): for $P \subset G$ a parabolic subgroup of a quasireductive supergroup with Levi part L ,

$$\sum_i (-1)^i Ch(H^i(G/P, \mathcal{L}(M^*))^*) = D \sum_{w \in W} \varepsilon(w) w \left(\frac{e^\rho Ch(M)}{\prod_{\alpha \in \Delta_{1,l}^+} (1 + e^{-\alpha})} \right),$$

where $D := \frac{D_0}{D_1}$, $D_0 = \prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2})$, $D_1 = \prod_{\alpha \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2})$, and the various Δ have the obvious composition (roots of \mathfrak{g} if no other index, roots corresponding to a subalgebra if the subalgebra appears as index).

Proposition 6. *Let $\nu = (a_m, \dots, a_1 | b_1, \dots, b_n) \in \Lambda_{m,n}^+$. Then one has:*

(1) $a > 0$, if $\exists i$ s.t. $a_{i+1} > a > a_i$,

$$\gamma^a(\mathcal{E}(\nu)) = (-1)^{m-i} \mathcal{E}(a_m, \dots, a_{i+1}, a, a_i, \dots, a_1 | b_1, \dots, b_n),$$

and $\gamma^a(\mathcal{E}(\nu)) = 0$ if $\exists i$, $a = a_i$.

(2) $a < 0$, if $\exists i$ s.t. $b_i < -a < b_{i+1}$,

$$\gamma^a(\mathcal{E}(\nu)) = (-1)^{n-i} \mathcal{E}(a_m, \dots, a_1 | b_1, \dots, b_i, -a, b_{i+1}, \dots, b_n),$$

and $\gamma^a(\mathcal{E}(\nu)) = 0$ if $\exists i$, $a = -b_i$.

(3) $b > 0$, if $\exists i$ s.t. $b = a_i$

$$\eta_b(\mathcal{E}(\nu)) = (-1)^{m-i} \mathcal{E}(a_m, \dots, a_{i+1}, a_{i-1}, \dots, a_1 | b_1, \dots, b_n)$$

if $b \neq a_i \forall i$, $\eta_b(\mathcal{E}(\nu)) = 0$.

(4) $b < 0$, if $\exists i$ s.t. $b = -b_i$

$$\eta_b(\mathcal{E}(\nu)) = (-1)^{n-i} \mathcal{E}(a_m, \dots, a_1 | b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$$

if $b \neq -b_i \forall i$, $\eta_b(\mathcal{E}(\nu)) = 0$.

Proof We will only prove (1), since (2) is analogous and then (3) and (4) follow by adjointness. Let us use [2], Theorem 1: one has, if M is a B -module,

$$\sum_{i,j} (-1)^{i+j} [H^i(G_{m,n}/P_{\underline{m},n}, \mathcal{L}(H^j(P_{\underline{m},n}/B, \mathcal{L}(M^*)))^*] = \sum_k (-1)^k [H^k(G_{m,n}/B, \mathcal{L}(M^*))^*].$$

We take for M the 1-dimensional representation \mathbb{C}_λ with $\lambda + \rho_B = (a, a_m, \dots, a_1 | b_1, \dots, b_n)$. Then, using the equation (4), and the definition of γ^a , we get

$$\gamma^a(\mathcal{E}(\nu)) = \mathcal{E}(\lambda) = (-1)^{m-i} (a_m, \dots, a_{i+1}, a, a_i, \dots, a_1 | b_1, \dots, b_n)$$

for the index i of the statement. Hence the proposition. \square

5. LINK WITH THE CLIFFORD ALGEBRA

Let us now interpret the map f of section 2 in terms of the functors described in the previous section. The proposition 6 has the following immediate corollary:

Corollary 2. *One has:*

$$\begin{aligned} f \circ \gamma^a &= v_a \circ f \text{ for } a > 0, \\ f \circ \gamma^a &= w_a \circ f \text{ for } a < 0, \\ f \circ \eta_b &= w_b \circ f \text{ for } b > 0, \\ f \circ \eta_b &= v_b \circ f \text{ for } b < 0, \end{aligned}$$

where v_a, w_b stand for the action on the Fock space of the corresponding elements of the Clifford algebra.

This gives us an action of the Clifford algebra on the Grothendieck group $\mathcal{K}(\mathcal{E})$.

Theorem 1. *The operators γ^a and η_b ($a, b \in \frac{1}{2} + \mathbb{Z}$) in the Grothendieck group \mathcal{K} satisfy the Clifford relations:*

$$\eta_a \eta_b + \eta_b \eta_a = 0, \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 0, \quad \gamma^a \eta_b + \eta_b \gamma^a = \delta_{a,b}.$$

Proof Let a and b be half-integers. We first show that

$$\eta_a \eta_b + \eta_b \eta_a = 0.$$

The arguments involved in the proof depend on the signs of a and b , we will take care of the cases $a, b > 0$ and $a > 0, b < 0$, leaving $a < 0, b < 0$ to the reader.

Assume first that $a > 0, b > 0$, let M be a $\mathfrak{g}_{m,n}$ module, we consider the following increasing chain of Lie superalgebras:

$$\mathfrak{g}_{m-2,n} \subset \mathfrak{p}_{\underline{m-1},n} \subset \mathfrak{g}_{m-1,n} \subset \mathfrak{p}_{\underline{m},n} \subset \mathfrak{g}_{m,n}.$$

Let \mathfrak{q} be the parabolic subalgebra with reductive part equal to the direct sum of $\mathfrak{g}_{m-2,n}$ and the two-dimensional center $Z_{\mathfrak{q}}$, and the nilradical $\mathfrak{r} = \mathfrak{r}_{\underline{m},n} + \mathfrak{r}_{\underline{m-1},n}$. Then using the Hochschild–Serre spectral sequence for the pair $\mathfrak{r}_{\underline{m-1},n} \subset \mathfrak{r}$ we obtain

$$\eta_a \eta_b [M] = \sum_i (-1)^i [\mathrm{Hom}_{Z_{\mathfrak{q}}}(\mathbb{C}_{(b-(n-m-1/2))\varepsilon_m + (a-(n-m+1/2))\varepsilon_{m-1}}, H^i(\mathfrak{r}, M))],$$

and

$$\eta_b \eta_a [M] = \sum_i (-1)^i [\mathrm{Hom}_{Z_{\mathfrak{q}}}(\mathbb{C}_{(a-(n-m-1/2))\varepsilon_m + (b-(n-m+1/2))\varepsilon_{m-1}}, H^i(\mathfrak{r}, M))].$$

Now we consider the one-dimensional root subalgebra $\mathfrak{s} := \mathfrak{g}_{\beta} \subset \mathfrak{r}$ for the root $\beta = \varepsilon_m - \varepsilon_{m-1}$. Note that \mathfrak{s} is the nilradical of a Borel subalgebra of the $\mathfrak{sl}(2)$ generated by \mathfrak{g}_{β} and $\mathfrak{g}_{-\beta}$. Hence by the Kostant theorem we have for any $\mathfrak{sl}(2)$ -module N

$$\begin{aligned} &[\mathrm{Hom}_{Z_{\mathfrak{q}}}(\mathbb{C}_{(a-(n-m-1/2))\varepsilon_m + (b-(n-m+1/2))\varepsilon_{m-1}}, H^p(\mathfrak{s}, N))] = \\ &[\mathrm{Hom}_{Z_{\mathfrak{q}}}(\mathbb{C}_{(b-(n-m-1/2))\varepsilon_m + (a-(n-m+1/2))\varepsilon_{m-1}}, H^q(\mathfrak{s}, N))] \end{aligned}$$

for $(p, q) = (0, 1)$ or $(1, 0)$.

Once again we apply the Hochschild–Serre spectral sequence for the pair $\mathfrak{s} \subset \mathfrak{r}$ to get

$$\begin{aligned}\eta_a \eta_b[M] &= \sum_i (-1)^{i+j} [\text{Hom}_{Z_q}(\mathbb{C}_{(b-(n-m-1/2))\varepsilon_m + (a-(n-m+1/2))\varepsilon_{m-1}}, H^i(\mathfrak{s}, \Lambda^j(\mathfrak{r}/\mathfrak{s})^* \otimes M))], \\ \eta_b \eta_a[M] &= \sum_i (-1)^{i+j} [\text{Hom}_{Z_q}(\mathbb{C}_{(a-(n-m-1/2))\varepsilon_m + (b-(n-m+1/2))\varepsilon_{m-1}}, H^i(\mathfrak{s}, \Lambda^j(\mathfrak{r}/\mathfrak{s})^* \otimes M))].\end{aligned}$$

This implies the relation.

Now let $a > 0, b < 0$. Let M be a $G_{m,n}$ -module. Set

$$\mathfrak{r} := \mathfrak{r}_{\underline{m},n} + \mathfrak{r}_{m-1,\underline{n}}, \quad \mathfrak{r}' := \mathfrak{r}_{m,\underline{n}} + \mathfrak{r}_{\underline{m},n-1}.$$

Let $Z \subset T_{m,n}$ be the centralizer of $\mathfrak{g}_{m-1,n-1}$. Using Hochschild–Serre spectral sequence we obtain

$$\eta_b \eta_a[M] = \sum_i (-1)^{m-1+i} [\text{Hom}_Z(\mathbb{C}_{(a-(m-n-1/2))\varepsilon_m - (b+n-m+1/2)\delta_n}, H^i(\mathfrak{r}, M))]$$

and

$$\eta_a \eta_b[M] = \sum_i (-1)^{m+i} [\text{Hom}_Z(\mathbb{C}_{(a-(m-n+1/2))\varepsilon_m - (b+n-m-1/2)\delta_n}, H^i(\mathfrak{r}', M))].$$

Let $\alpha = \varepsilon_m - \delta_n$. Consider the root subalgebras $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \subset \mathfrak{g}_{m,n}$. Note that $\mathfrak{s} := \mathfrak{r} \cap \mathfrak{r}'$ is an ideal of codimension 1 in both \mathfrak{r} and \mathfrak{r}' and that

$$\mathfrak{r} = \mathfrak{s} + \mathfrak{g}_\alpha, \quad \mathfrak{r}' = \mathfrak{s} + \mathfrak{g}_{-\alpha}.$$

Therefore by Hochschild–Serre spectral sequence we have

$$\begin{aligned}\eta_b \eta_a &= \sum_{i,j} (-1)^{i+j+m-1} [\text{Hom}_Z(\mathbb{C}_{(a-(m-n-1/2))\varepsilon_m - (b+n-m+1/2)\delta_n}, \Lambda^j(\mathfrak{g}_{-\alpha}) \otimes H^i(\mathfrak{s}, M))], \\ \eta_a \eta_b[M] &= \sum_{i,j} (-1)^{i+j+m} [\text{Hom}_Z(\mathbb{C}_{(a-(m-n+1/2))\varepsilon_m - (b+n-m-1/2)\delta_n}, \Lambda^j(\mathfrak{g}_\alpha) \otimes H^i(\mathfrak{s}, M))].\end{aligned}$$

Taking into account that

$$\sum_j (-1)^j Ch(\Lambda^j(\mathfrak{g}_\alpha)) = \frac{1}{1+e^\alpha} = \frac{e^{-\alpha}}{1+e^{-\alpha}} = e^{-\alpha} \left(\sum_j (-1)^j Ch(\Lambda^j(\mathfrak{g}_{-\alpha})) \right)$$

we obtain

$$\sum_j (-1)^j Ch(\Lambda^j(\mathfrak{g}_\alpha) \otimes H^i(\mathfrak{s}, M)) = e^{-\alpha} \left(\sum_j (-1)^j Ch(\Lambda^j(\mathfrak{g}_{-\alpha}) \otimes H^i(\mathfrak{s}, M)) \right).$$

Therefore

$$Ch(\eta_a \eta_b[M]) = -Ch(\eta_b \eta_a[M]),$$

which proves the relation.

Note that the relation

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 0.$$

follows from the relation for η_a, η_b by Proposition 5.

Let us now show that if $a > 0$ and $b < 0$, then

$$\gamma^a \eta_b + \eta_b \gamma^a = 0.$$

The case $a < 0$ and $b > 0$ is similar and we leave it to the reader.

One should keep in mind the following diagram

$$\begin{array}{ccc} & \eta_b & \\ \mathcal{F}_{m,n} & \rightarrow & \mathcal{F}_{m,n-1} \\ \gamma^a \downarrow & & \downarrow \gamma^a \\ \mathcal{F}_{m+1,n} & \rightarrow & \mathcal{F}_{m+1,n-1} \\ & \eta_b & \end{array}$$

because we follow it to keep tracks of the weights.

Let us denote ChM_γ the character of $\text{Hom}_Z(\mathbb{C}_\gamma, M)$. Then one has:

$$\begin{aligned} & \gamma^a \eta_b Ch(M) = \\ = & \sum_{i,j} (-1)^{i+j+m} Ch(\Gamma_j(G_{m+1,n-1}/P_{\underline{m+1},n-1}, (\mathbb{C}_{(a-(m-n+1/2))\varepsilon_{m+1}} \boxtimes \Lambda^i(\mathfrak{r}_{m,\underline{n}}^*) \otimes M)_{(-b-(n-m-1/2))\delta_n})), \end{aligned}$$

$$\begin{aligned} & \eta_b \gamma^a Ch(M) = \\ = & \sum_{i,j} (-1)^{i+j+m+1} Ch((\Lambda^i(\mathfrak{r}_{m+1,\underline{n}}^*) \otimes \Gamma_j(G_{m+1,n}/P_{\underline{m+1},n}, \mathbb{C}_{(a-(m-n-1/2))\varepsilon_{m+1}} \boxtimes M))_{(-b-(n-m-3/2))\delta_n}). \end{aligned}$$

We use Proposition 3. For any $G_{m,k}$ -module N , the following holds:

$$\sum_j (-1)^j \Gamma_j(G_{m+1,k}/P_{\underline{m+1},k}, N) = \sum_j (-1)^j \Gamma_j(G_{m+1,0}/P_{\underline{m+1},0}, N \otimes S^\bullet((\mathfrak{r}_{\underline{m+1},k}^*)_{\bar{1}}).$$

Then if we set:

$$X := \sum_i (-1)^i \Lambda^i(\mathfrak{r}_{m,\underline{n}}^*) \otimes S^\bullet((\mathfrak{r}_{\underline{m+1},n-1}^*)_{\bar{1}})$$

and

$$Y := \sum_i (-1)^i \Lambda^i(\mathfrak{r}_{m+1,\underline{n}}^*) \otimes S^\bullet((\mathfrak{r}_{\underline{m+1},n}^*)_{\bar{1}}),$$

we get

$$\begin{aligned} & \gamma^a \eta_b (Ch(M)) = \\ = & \sum_j (-1)^{j+m} Ch(\Gamma_j(G_{m+1,0}/P_{\underline{m+1},0}, \mathbb{C}_{(a-(m-n+1/2))\varepsilon_{m+1}} \boxtimes X \otimes M)_{(-b-(n-m-1/2))\delta_n}) \end{aligned}$$

and

$$\begin{aligned} & \eta_b \gamma^a (Ch(M)) = \\ = & \sum_j (-1)^{j+m+1} Ch(\Gamma_j(G_{m+1,0}/P_{\underline{m+1},0}, \mathbb{C}_{(a-(m-n-1/2))\varepsilon_{m+1}} \boxtimes Y \otimes M)_{(-b-(n-m-3/2))\delta_n}). \end{aligned}$$

Next we compute the quotient $Ch(X)/Ch(Y)$. One has

$$Ch(X) = \frac{(1 - e^{-2\delta_n}) \prod_{i=1}^{n-1} (1 - e^{-\delta_n \pm \delta_i}) \prod_{j=1}^{n-1} (1 + e^{\pm \delta_j - \varepsilon_{m+1}})}{(1 + e^{-\delta_n}) \prod_{j=1}^m (1 + e^{-\delta_n \pm \varepsilon_j})},$$

$$Ch(Y) = \frac{(1 - e^{-2\delta_n}) \prod_{i=1}^{n-1} (1 - e^{-\delta_n \pm \delta_i}) \prod_{j=1}^n (1 + e^{\pm \delta_j - \varepsilon_{m+1}})}{(1 + e^{-\delta_n}) \prod_{j=1}^{m+1} (1 + e^{-\delta_n \pm \varepsilon_j})},$$

and the quotient turns out to be

$$Ch(X)/Ch(Y) = e^{\varepsilon_{m+1} - \delta_n}.$$

The result follows.

Let us show finally that for $a, b > 0$ one has

$$\gamma^a \eta_b + \eta_b \gamma^a = \delta_{a,b}$$

where $\delta_{a,b}$ stands for the Kronecker symbol. The proof we provide lacks functoriality at the moment, but we intend to improve it.

Let $R : \mathcal{F}_{m,n} \rightarrow \mathcal{F}_{m,0}$ be the restriction functor and denote by the same letter the corresponding map of the Grothendieck groups. Then it follows from Proposition 3 that for any $M \in \mathcal{F}_{m,n}$,

$$R(\gamma^a[M]) = \gamma^{a+n}([S^\bullet((\mathfrak{r}_{\underline{m+1},n}^*)_{\bar{1}})]R[M]).$$

On the other hand, for any Lie superalgebra \mathfrak{r} and \mathfrak{r} -module M we have

$$\sum_i (-1)^i [H^i(\mathfrak{r}, M)] = \sum_{k,l} (-1)^{k+l} [\Lambda^l(\mathfrak{r}_{\bar{1}}^*)] [H^k(\mathfrak{r}_{\bar{0}}, M)].$$

Therefore

$$R(\eta_a[M]) = \sum_k (-1)^k \eta_{a+n}([\Lambda^k((\mathfrak{r}_{\underline{m},n}^*)_{\bar{1}})]R[M]).$$

Therefore for $M \in \mathcal{F}_{m,n}$ we have

$$R(\gamma^a \eta_b[M]) = \sum_k (-1)^k \gamma^{n+a} ([S^\bullet((\mathfrak{r}_{\underline{m},n}^*)_{\bar{1}})] \eta_{b+n}([\Lambda^k((\mathfrak{r}_{\underline{m},n}^*)_{\bar{1}})]R[M])),$$

$$R(\eta_b \gamma^a[M]) = \sum_k (-1)^k \eta_{b+n} \left([\Lambda^k((\mathfrak{r}_{\underline{m+1},n}^*)_{\bar{1}})] \gamma^{n+a} ([S^\bullet((\mathfrak{r}_{\underline{m+1},n}^*)_{\bar{1}})]R[M]) \right).$$

Let us denote by U the standard representation of $\mathfrak{sp}(2n) \subset \mathfrak{osp}(2m+1, 2n)$ and consider it as purely odd superspace. Then

$$Ch((\mathfrak{r}_{\underline{m},n}^*)_{\bar{1}}) = e^{-\varepsilon_m} Ch(U).$$

Therefore the above expressions can be rewritten in the form

$$R(\gamma^a \eta_b[M]) = \sum_{k,l} (-1)^k \gamma^{n+a-l} ([S^l(U)] \eta_{b+n+k}([\Lambda^k(U)]R[M])),$$

$$R(\eta_b \gamma^a[M]) = \sum_k (-1)^k \eta_{b+n+k} ([\Lambda^k(U)] \gamma^{n+a-l} ([S^l(U)]R[M])).$$

Now we note that the action of $G_{m,0}$ on U is trivial, hence multiplication with its exterior and symmetric powers commute with γ^a and η_b . Thus, we have

$$R(\gamma^a \eta_b [M]) = \sum_{k,l} (-1)^k [S^l(U)] [\Lambda^k(U)] \gamma^{n+a-l} \eta_{b+n+k} (R[M]),$$

$$R(\eta_b \gamma^a [M]) = \sum_k (-1)^k ([\Lambda^k(U)] [S^l(U)] \eta_{b+n+k} \gamma^{n+a-l} (R[M])).$$

Since $\mathcal{F}_{m,0}$ is the category of representations of a purely even reductive group, we have $\mathcal{K}(\mathcal{E})_{m,0} = \mathcal{K}(L)_{m,0}$. Therefore Proposition 6 implies that for any $N \in \mathcal{F}_{m,0}$

$$\gamma^a \eta_b [N] + \eta_b \gamma^a [N] = \delta_{a,b} [N].$$

Hence,

$$(\gamma^a \eta_b + \eta_b \gamma^a)(R[M]) = \sum_{l,k} (-1)^k [S^l(U)] [\Lambda^k(U)] \delta_{a+n-k, b+n+l} R[M],$$

and hence

$$(\gamma^a \eta_b + \eta_b \gamma^a)(R[M]) = \sum_{k+l=a-b} (-1)^k [S^l(U)] [\Lambda^k(U)] R[M].$$

Since the Koszul complex is acyclic except in the zero degree we have the identity

$$\sum_{k+l=p} (-1)^k [\Lambda^k(U)] [S^l(U)] = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Hence, the sum we compute has only one non-zero term, namely we get:

$$(\gamma^a \eta_b + \eta_b \gamma^a)(R[M]) = \delta_{a,b} R[M].$$

Since the map R is injective this proves the result for $\gamma^a \eta_b + \eta_b \gamma^a$, $a, b > 0$. The case $a, b < 0$ is similar and we leave it to the reader. \square

6. TRANSLATION FUNCTORS

We would like to link this approach with the results on translation functors in [3].

Recall the Lie algebra $\mathfrak{gl}(\infty)$ which is embedded in $Cl(V \oplus W)$ as the span of $v_a w_b$, $a, b \in \frac{1}{2} + \mathbb{Z}$. The subalgebra $\mathfrak{gl}(\frac{\infty}{2})$ is generated by $v_a w_b + v_{-a} w_{-b}$, $a, b \in \frac{1}{2} + \mathbb{N}$.

Inside the Fock space \mathbf{F} , we consider the subspace $\mathbf{F}_{m,n}$ which is the image of $\mathcal{K}(\mathcal{E})_{m,n}$ under the map f , defined at the end of section 2.

Remark 1. The space $\mathbf{F}_{m,n}$ is stable under the action of $\mathfrak{gl}(\frac{\infty}{2})$. Furthermore, it is not difficult to see that $\mathbf{F}_{m,n}$ is isomorphic to $\Lambda^m(V_+) \otimes \Lambda^n(W_+)$ as an $\mathfrak{sl}(\frac{\infty}{2})$ -module, where V_+ and W_+ are respectively the standard and costandard module of $\mathfrak{gl}(\frac{\infty}{2})$.

Consider the Cartan subalgebra \mathfrak{t} of $\mathfrak{gl}(\frac{\infty}{2})$ with basis $t_a := v_a w_a + v_{-a} w_{-a}$ for all $a \in \frac{1}{2} + \mathbb{N}$, then \mathbf{F} is a semi-simple \mathfrak{t} -module. We denote by ω the \mathfrak{t} -weight of the vacuum vector: $\omega(t_a) = 1$ for all $a \in \frac{1}{2} + \mathbb{N}$. Let $\beta_a \in \mathfrak{t}^*$ be such that $\beta_a(t_b) = \delta_{a,b}$. If $\lambda = (a_m, \dots, a_1 | b_1, \dots, b_n)$, then the \mathfrak{t} -weight of $f(\mathcal{E}(\lambda))$ equals

$$\beta(\lambda) := \omega + \beta_{a_1} + \dots + \beta_{a_m} - \beta_{b_1} - \dots - \beta_{b_n}.$$

Lemma 2. *Let $\mathcal{E}(\lambda), \mathcal{E}(\mu) \in \mathcal{K}(\mathcal{E})_{m,n}$. Then $\mathcal{E}(\lambda)$ and $\mathcal{E}(\mu)$ are in the same block of $\mathcal{F}_{m,n}$ if and only if the \mathfrak{t} -weights of $f(\mathcal{E}(\lambda))$ and $f(\mathcal{E}(\mu))$ coincide.*

Proof. The statement follows from the remark 1 after comparing with the weights denoted by $\gamma(\lambda)$ in [3] (we do not keep this notation here because we have introduced a γ^a which is not related). The relation between those \mathfrak{t} -weights is $\beta(\lambda) = \omega + \gamma(\lambda)$. \square

Consider now the Chevalley generators of $\mathfrak{gl}(\frac{\infty}{2})$, $E_{a,a+1}$ and $E_{a+1,a}$ for all $a \in \frac{1}{2} + \mathbb{N}$. As it was shown in [3], the categorification of the action of these generators in $\Lambda^m(V_+) \otimes \Lambda^n(W_+)$ is given by the translation functors:

$$T_{a+1,a}(M) := (M \otimes E)_{\beta + \beta_{a+1} - \beta_a}, \quad T_{a,a+1}(M) := (M \otimes E)_{\beta + \beta_a - \beta_{a+1}},$$

where E is the standard $\mathfrak{g}_{m,n}$ -module, we assume that the $\mathfrak{g}_{m,n}$ -module- M belongs to the block corresponding to the \mathfrak{t} -weight β , and by $(N)_{\beta'}$ we denote the projection of the $\mathfrak{g}_{m,n}$ -module N onto the block corresponding to the \mathfrak{t} -weight β' . By abuse of notations we denote also by $T_{a+1,a}$ and $T_{a,a+1}$ the corresponding linear operators in $\mathcal{K}(\mathcal{E})_{m,n}$.

The following statement is an immediate consequence of the remark 1 and Lemma 4 in [3].

Proposition 7. *For all $a \in \frac{1}{2} + \mathbb{N}$ we have*

$$f \circ T_{a+1,a} = E_{a+1,a} \circ f, \quad f \circ T_{a,a+1} = E_{a,a+1} \circ f.$$

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